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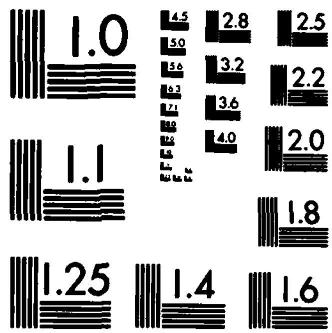
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MRC Technical Summary Report #2417

TRAVELING WAVE SOLUTIONS OF A MULTISTABLE REACTION-DIFFUSION EQUATION

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September 1982

(Received June 8, 1982)

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National Science Foundation
Washington, D. C. 20550

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

A new method for proving the existence of traveling wave solutions for equations of the form

$$u_t = u_{xx} + F'(u)$$

is presented. It is assumed that F is sufficiently smooth, $\lim_{|u|\to\infty} F(u) = -\infty$, F has only nondegenerate critical points, and if A and B are distinct critical points of F then $F(A) \neq F(B)$. The results describe when, for a given function F, there must exist zero, exactly one, a finite number, or an infinite number of waves which connect two fixed, stable rest points.

AMS(MOS) Subject Classification: 35K55

Key Words: Reaction-diffusion Equation, Traveling Wave Solution

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS80-17158.

SIGNIFICANCE AND EXPLANATION

The equation considered here has been considered as a model for a variety of physical phenomena including population genetics and nerve conduction. Of primary interest is the eventual behavior of solutions of this equation. One expects the solution eventually to look like a traveling wave solution; that is, one which moves with constant shape and velocity. In this paper we determine all of the traveling wave solutions of the equation, showing there are situations when there exist an infinite number of traveling wave solutions.

TRAVELING WAVE SOLUTIONS OF MULTISTABLE REACTION-DIFFUSION EQUATIONS

David Terman

§1. Introduction

Consider the equation

$$(1.1) \qquad u_t = u_{xx} + F'(u)$$

which arises in various branches of mathematical biology including population genetics, ecology, and nerve conduction (see [1],[3]). We assume throughout that F satisfies:

$$(1.2) \qquad \begin{array}{ll} \text{(a)} & F \in C^1(\mathbb{R}), \\ \text{(b)} & \lim\limits_{|u|\to\infty} F(u) = -\infty, \\ \text{(c)} & \text{every critical point of } F \text{ is nondegenerate,} \\ \text{(d)} & \text{if } A \text{ and } B \text{ are distinct critical points of } F, \text{ then} \\ & F(A) \neq F(B). \end{array}$$

We are interested in finding traveling wave solutions of (1.1). These are nonconstant, bounded solutions of the form $u(x,t) = U(z)$, $z = x + \theta t$. If $U(z)$ is a traveling wave solution of (1.1) then U satisfies the first order system of ordinary differential equations:

$$(1.3) \qquad \begin{aligned} U' &= V \\ V' &= \theta V - F'(U). \end{aligned}$$

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For boundary conditions we take

$$(1.4) \qquad \lim_{z \to -\infty} U(z) = A \quad \text{and} \quad \lim_{z \to +\infty} U(z) = B$$

where A and B are stable rest points of equation (1.1). Note that the stable rest points of equation (1.1) correspond to those values of U for which F assumes a local maximum. If (U,U') is a solution of equations (1.3) and (1.4) we shall sometimes say that "U connects $A \to B$". We shall also sometimes call a traveling wave solution a "connection". In this paper we develop a technique to determine, for a given function F and two stable rest points A and B, how many different waves connect $A \to B$. We shall see that there may exist zero, a finite number, or a countably infinite numbers of such waves.

The case when $F(u)$ has exactly two local maxima has been considered by a number of authors (see [1] for references). Some work on the multistable case is in the paper of Fife and McLeod [3]. If F has just two local maxima then $F'(u)$ has the familiar cubic shape. In this case there exists a unique wave with positive speed which connects the stable rest points.

It follows from (1.3) that if $U(z)$ is a traveling wave solution, then

$$(1.5) \qquad \frac{d}{dz}\{\frac{1}{2} V^2 + F(u)\} = \theta V^2.$$

We assume throughout that the speed, θ, is positive. An immediate consequence of this assumption and (1.5) is that if U connects $A \to B$, then $F(A) < F(B)$. Note that if $u(x,t)$ is a traveling wave solution moving to the left with positive speed θ, then $u(-x,t)$ is a traveling wave solution moving to the right with negative speed, $-\theta$.

The (U,V) phase plane is the natural place to study the solutions of system (1.3). In the phase plane, stable rest points of equation (1.1) correspond to saddles, while traveling wave solutions correspond to trajectories which "connect" the saddles. One can only expect saddle-saddle connections to exist for special values of the speed θ. The difficulty with phase plane analysis is that the phase planes become much too complicated for a general function F. For example, even when F has only three local maxima there may exist an infinite number of traveling wave solutions.

To illustrate the approach we take, let F be as shown in Figure 1.

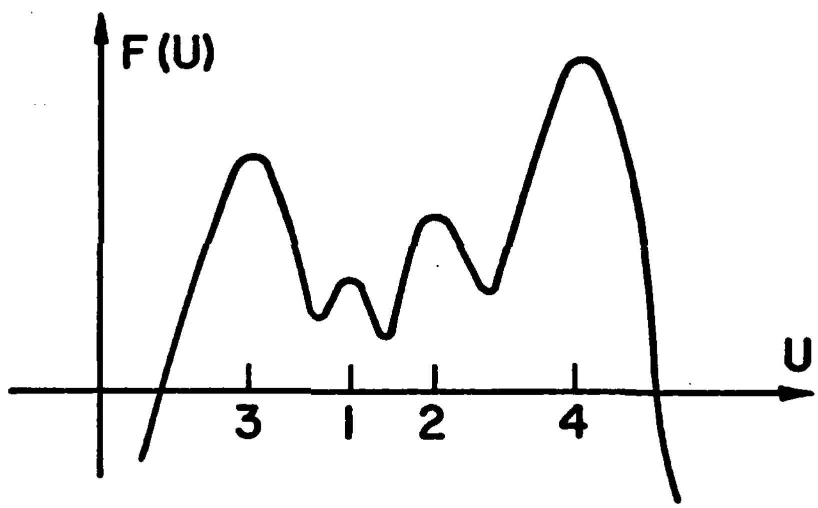

Figure 1. Notice that the local maxima of F are ordered according to the height determined by F.

We suppose that there are four values of U, say A, B, C, and D, where F assumes a local maximum. Assume that $A < B < C < D$ and $F(D) > F(A) > F(C) > F(B)$. Notice that in Figure 1 we ordered the four critical points according to the height determined by F. That is, we set $A = 3$, $B = 1$, $C = 2$, and $D = 4$. Unless stated otherwise we shall always order the stable rest points, or local maxima of F, in this manner. Our description of how many traveling waves exist shall be in terms of this ordering. Two functions

which satisfy the conditions (1.2) are said to be in the same equivalence class if they have the same ordering of their local maxima. Hence, given a positive integer n, each permutation of the integers $1,2,\ldots,n$ determines a unique equivalence class of functions.

For a given function F there exists a speed θ_0 such that no saddle-saddle connections exist for $\theta > \theta_0$. It is not very hard to determine what the phase plane must look like for $\theta > \theta_0$. If F is as shown in Figure 1 then the phase plane for θ sufficiently large looks, qualitatively, like what is illustrated in Figure 2A.

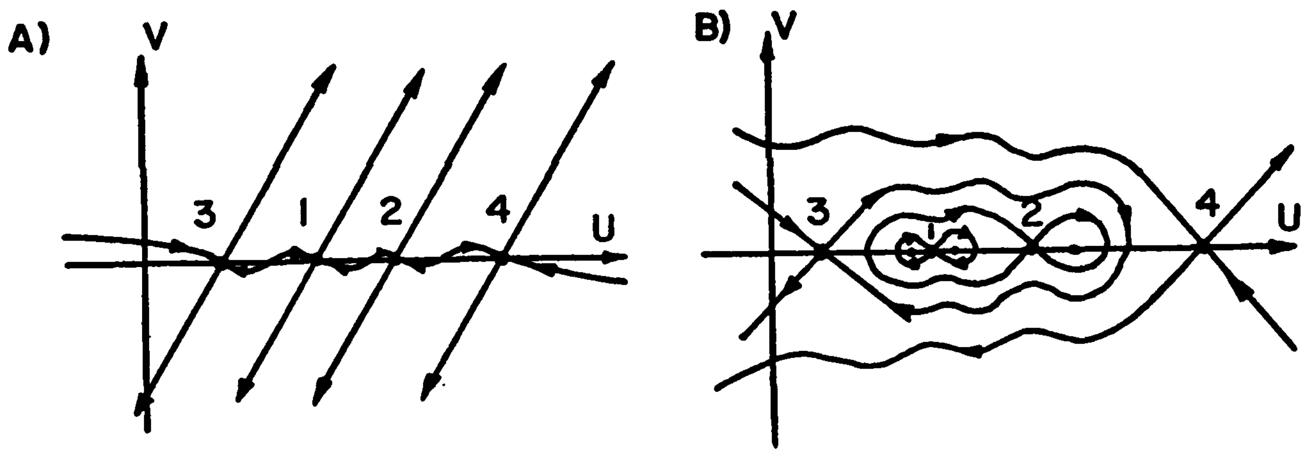

Figure 2. Phase planes for System 1.2 where F is as shown in Figure 1. In (A) the speed, θ, is very large, and in (B), $\theta = 0$.

As θ approaches $+\infty$ the unstable manifolds (trajectories which approach the saddles in backwards time) become more and more vertical while the stable manifolds (trajectories which approach the saddles in forward time) become more horizontal. In Figure 2B we show the phase plane for $\theta = 0$.

The basic approach is to begin at $\theta = \theta_0$ and then start decreasing θ. By comparing the phase plane for $\theta = \theta_0$ with that for $\theta = 0$ one is able to

determine all the possibilities for the fastest wave. For the function F shown in Figure 1 these possibilites are the $1 \to 3$, $1 \to 2$, and $2 \to 4$ connections. Of course, which one of these is the fastest wave depends on the specific function F. Let us suppose that for a particular F the fastest wave connects $1 \to 3$. Then the qualitative features of the phase planes change after the $1 \to 3$ connection (see Figure 3). We can then decrease θ further and determine, by comparing the new phase plane with the phase plane at $\theta = 0$, all the possibilites for the next fastest connection. These possibilities are the $2 \to 3$, $1 \to 2$, and $2 \to 4$ connections. A similar analysis can be done if the fastest wave was either the $1 \to 2$ or the $2 \to 4$ connection. In this manner we can construct a directed graph as shown in Figure 3. This graph illustrates all the possible orderings of which connections can take place. To each equivalence class of functions there corresponds such a directed graph. To each specific function there corresponds a path in the directed graph determined by its equivalence class. This path first shows the fastest wave, then the second fastest wave, etc... Our immediate goal is to be able to construct, and understand, these graphs.

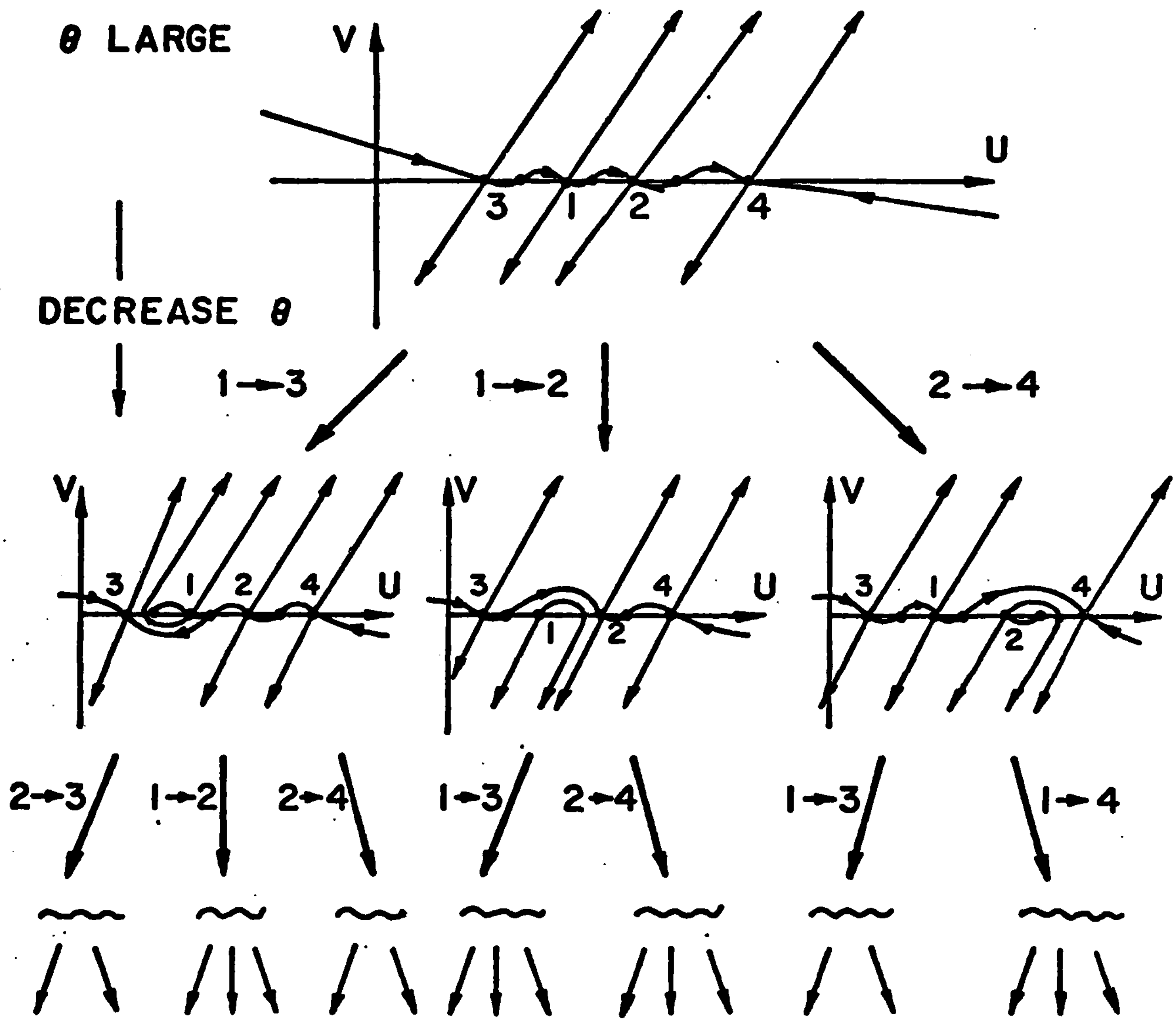

Figure 3. The directed graph for the equivalence class of functions shown in Figure 1.

One approach to constructing these graphs would be to draw a lot of phase planes. This would be very tedious, if not impossible, since the phase planes become very complicated. What is needed is a way to quantify the essential

information contained in each phase plane. That is, we need a way to translate each phase plane into an array of numbers. Then, by just looking at the array of numbers, we can hopefully determine what the possibilities are for the next fastest wave. After a connection takes place the qualitative, or topological, features of the phase planes change. This implies that the array of numbers also changes. Hence, we need an algorithm which tells us how to change the array of numbers after a particular connection has occurred.

In section 2 we describe how to

(1.6)
(a) assign an array of numbers to each phase plane,
(b) determine what the possibilities are for the next fastest wave by just looking at the array,
(c) change the array after a connection has taken place.

With these three operations we will be able to construct all of the directed graphs. In section 3 we present some applications of the technique just described. In section 4 we present further applications and give a complete description of the tristable equation. A complete proof of all the results presented in this paper may be found in [4].

Acknowledgement: The author would like to thank Professor Charles Conley for many enlightening discussions of the problem.

§2. Directed Graphs

We illustrate how to do the three operations described in (1.6) with an example. Suppose we are given a function F which is in the equivalance class of functions illustrated in Figure 1. Furthermore, suppose we know that at $\theta = \theta_0$ the phase plane is as shown in Figure 4. We wish to determine what the possibilities are for the next fastest wave. Only the unstable manifolds are shown in Figure 4 because, in order to determine what the possibilities are for next fastest wave, that is all we need to know.

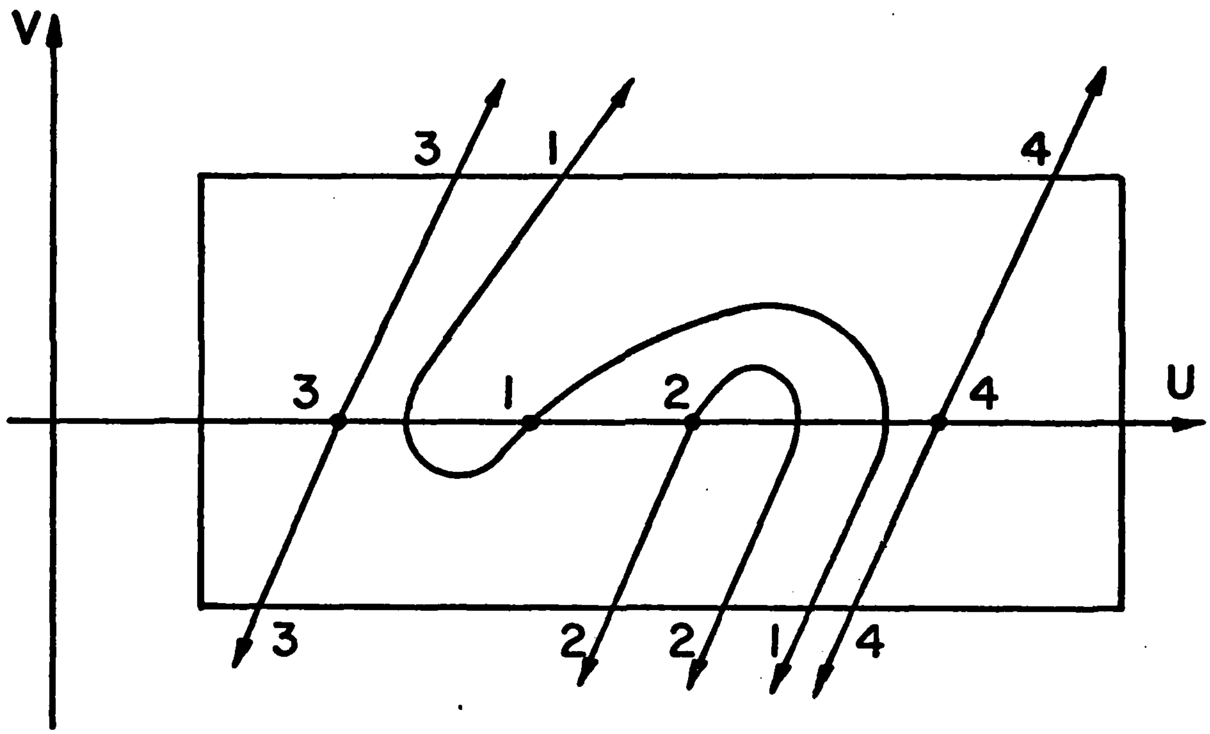

Figure 4. The array of numbers associated with this phase plane is $\frac{314}{32214}$.

The first step in assigning an array of numbers to the phase plane is to draw a big rectangle, R, around the rest points. Since the set of bounded trajectories is compact (see Conley [2]), R can be chosen so that all the connections, for all values of θ, lie inside of it.

The next step is to locate those points on the boundary of $\mathcal{R}$ which lie on one of the unstable manifolds. By analyzing system (1.3) one finds that $\mathcal{R}$ can be chosen so that these points either lie on the top side or the bottom side of $\mathcal{R}$. To each one of these points there corresponds a number; that number being the saddle on whose unstable manifold the point lies. We now have two lists of numbers; one corresponding to the top side of $\mathcal{R}$, and the other to the bottom side. For the example shown in Figure 4 we have the two lists {3,1,4} and {3,2,2,1,4}. Combining the two lists we obtain:

$$\frac{314}{32214}\,. \tag{2.1}$$

This is the desired array of numbers!

We claim that the array shown in (2.1) determines all the possibilities for the next fastest traveling wave solution. We explain how these possibilities are realized with the following proposition.

Proposition 1: Suppose that for a given function F we have, at some speed $\theta = \theta_0$, the ordering $\frac{T_1\ T_2\ \cdots\ T_m}{B_1\ B_2\ \cdots\ B_n}$. If, for some k, $T_k < T_{k+1}$ then there exists a connection $T_k \to T_{k+1}$ for some $\theta \in (0,\theta_0)$. If, for some k, $B_k > B_{k+1}$ then there exists a connection $B_{k+1} \to B_k$ for some $\theta \in (0,\theta_0)$. These give all the possibilities for the next fastest wave. That is, if the next fastest wave corresponds to an $A \to B$ connection then there exists an integer k such that either $A = T_k$ and $B = T_{k+1}$, or $A = B_{k+1}$ and $B = B_k$.

Let us apply this proposition to the phase plane shown in Figure 4. Looking at the array (2.1) and applying Proposition 1 we find that the

possibilities for the next fastest wave are the $1 \to 4$, $2 \to 3$, and $1 \to 2$ connections. Note that the $1 \to 3$ connection may or may not exist for some $\theta \in (0, \theta_0)$, but it cannot be the next fastest connection. Furthermore, the $1 \to 4$, $2 \to 3$, and $1 \to 2$ connections must all exist for some speeds less than θ_0.

We now need an algorithm which tells us how the array changes after a connection has taken place. This algorithm is described in the following proposition. For this proposition we assume that the array is known for some value of the speed, say $\theta = \theta_0$. We also assume that the next fastest connection is an $A \to B$ connection. Note that there must be two B's in the array since to each saddle there corresponds two unstable directions. Of course, there are two A's also, but the 'other A' will play no role. In the proposition we consider two cases depending on whether the 'other B' is on the top or the bottom of the array.

<u>Proposition 2.</u> If the other B is on the top then after the $A \to B$ connection everything in the array remains exactly the same except the A is moved to the immediate right of the other B. If the other B is on the bottom then after the $A \to B$ connection everything in the array remains exactly the same except the A is moved to the immediate left of the other B.

Here are two examples of what may happen:

(a) $$\frac{\cdot\cdot AB\cdot\cdot B\cdot\cdot}{\cdot\cdot\cdot\cdot} \xrightarrow{A \to B} \frac{\cdot\cdot B\cdot\cdot BA\cdot\cdot}{\cdot\cdot\cdot\cdot}$$

(b) $$\frac{\cdot\cdot AB\cdot\cdot\cdot}{\cdot\cdot B\cdot\cdot} \xrightarrow{A \to B} \frac{\cdot\cdot B\cdot\cdot\cdot}{\cdot\cdot AB\cdot\cdot}$$

Applying this proposition to the phase plane shown in Figure 4 we have the following portion of the graph:

$$
\begin{tikzcd}
 & \frac{314}{32214} \arrow[dl, "1\to4"'] \arrow[d, "2\to3"] \arrow[dr, "1\to2"] & \\
\frac{34}{322114} & \frac{3214}{3214} & \frac{314}{31224}
\end{tikzcd}
$$

In order to complete the description of how to construct the directed graph it is necessary to explain how one starts the graph when the speed is very large. It is not hard to show that if the ordering of the saddles is $A_1, A_2, \ldots, A_n$ then the graph begins with the array.

$$\frac{A_1\ A_2\ \cdots\ A_n}{A_1\ A_2\ \cdots\ A_n}\ .$$

This is because when θ is very large, then in the phase plane the unstable trajectories are nearly verticle. For example, if F is in the equivalence class of functions shown in Figure 1 then the graph starts with the array $\frac{3124}{3124}$. The rest of the directed graph is shown in Figure 5.

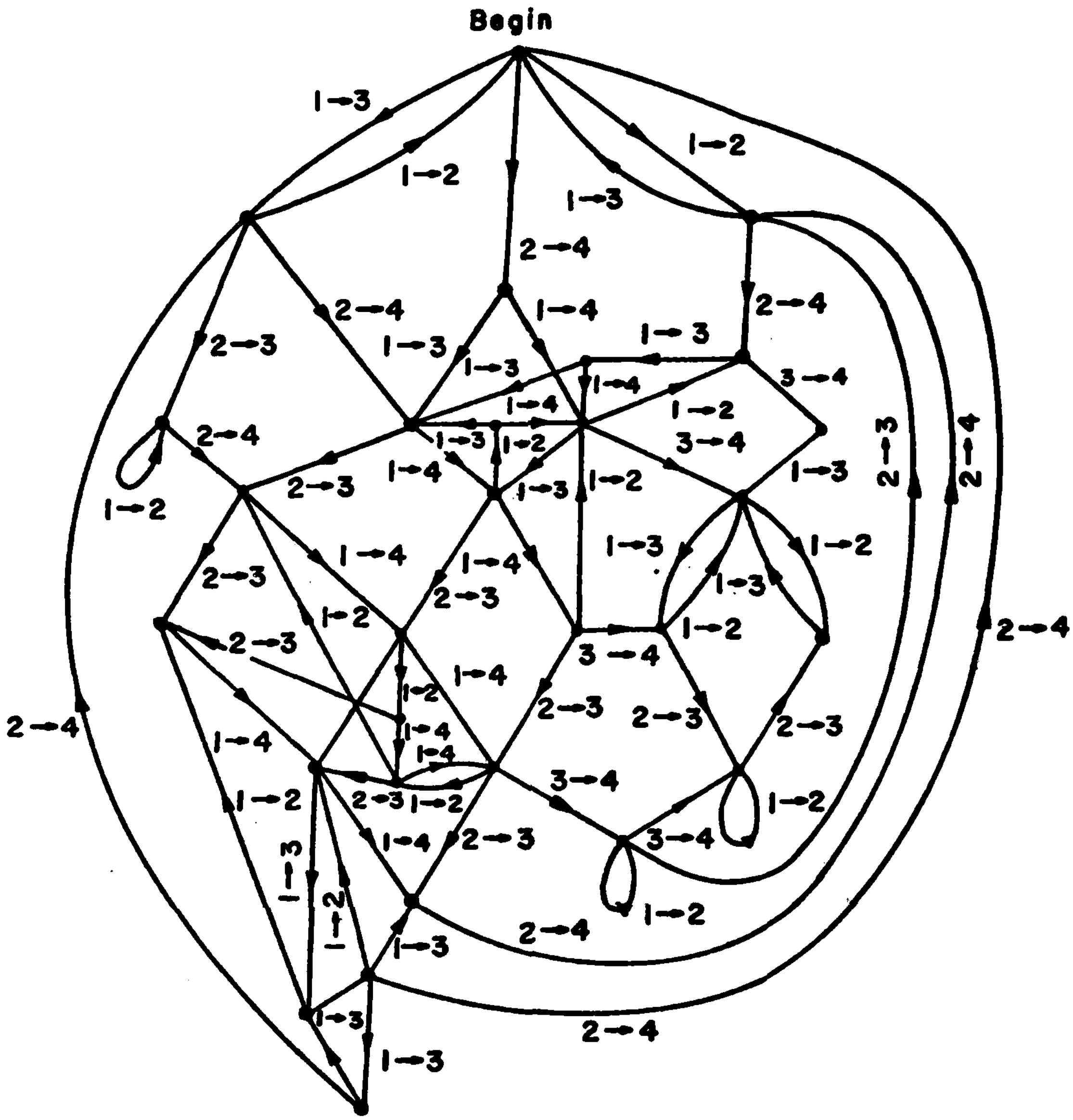

Figure 5. The complete directed graph associated with the equivalence class of functions shown in Figure 1.

§3. Applications

Suppose that F is in the equivalence class of functions illustrated by Figure 6.

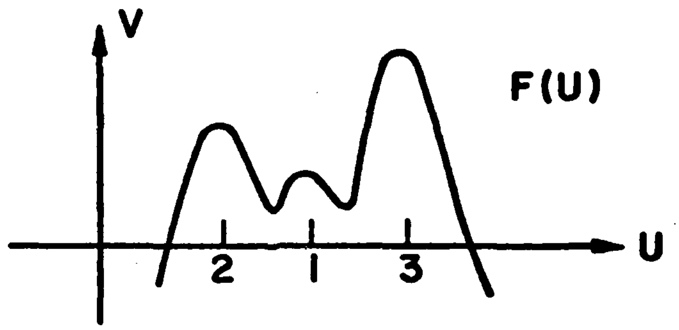

Figure 6. For the equivalence class of functions shown here there exists an infinite number of $1 \to 2$ connections, a finite number of $1 \to 3$ connections and precisely one, $2 \to 3$ connection.

It follows from a simple shooting argument that there exists a unique wave connecting $2 \to 3$. We use this fact and the ideas described in the previous section to demonstrate that there exists a finite number of waves connecting $1 \to 3$ and an infinite number of waves connecting $1 \to 2$.

To prove these results we consider the graph associated with this equivalence class of functions. The graph begins when θ is very large with the array $\frac{213}{213}$. Then, using Propositions 1 and 2 we find that the desired graph can be represented as simply:

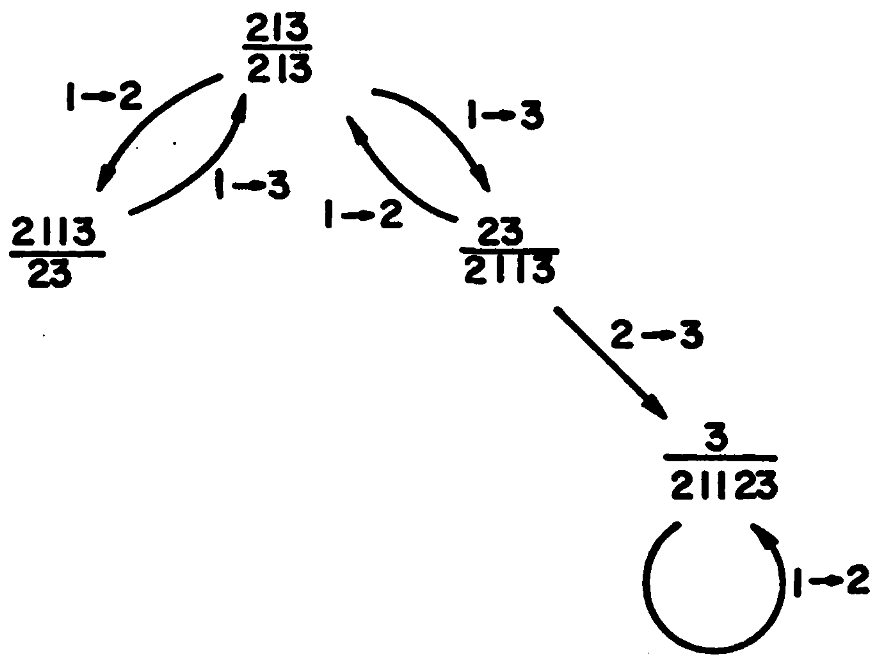

Figure 7. The directed graph associated with the equivalence class of functions shown in Figure 6.

Each function in the equivalence class shown in Figure 6 determines a path in the above directed graph. This path represents the order, starting with the fastest, in which the various connections take place. Until now all we know about this path is that it must obey the arrows in the directed graph, and eventually a $2 \to 3$ connection must take place. However, there is only one $2 \to 3$ connection in the graph. After the $2 \to 3$ connection we have the array $\frac{3}{21123}$. Proposition 1 implies that the next connection has to be a $1 \to 2$ connection. This, however, leaves the array unchanged, and is, therefore, represented by the loop in Figure 7. The phase plane, of course, changes after the $1 \to 2$ connection, but not the array. Therefore, after the $2 \to 3$ connection we are forced to go around this loop an infinite number of times, proving that there must exist an infinite number of $1 \to 2$ connections.

A separate argument shows that there can be at most a finite number of $1 \to 3$ connections with speeds greater than the speed of the $1 \to 2$ connection. There cannot be any $1 \to 3$ connections with slower speeds since then we're caught up in the $1 \to 2$ loop. The directed graph shows, however, that there must be at least one $1 \to 3$ connection, so this completes the proof.

Our second application is a generalization of the first one. Here it is assumed that $\bar{A} = (A,0)$ and $\bar{B} = (B,0)$ are saddles in the (U,V) phase plane with $A < B$ and $F(A) < F(B)$. Furthermore, if $\bar{C} = (C,0)$ is any other saddle such that $A < C < B$, then $F(C) < F(A)$. Let $\bar{D}_1, \bar{D}_2, \ldots, \bar{D}_n$ be the saddles which satisfy $A < D_k < B$ for each k. Here $\bar{D}_k \equiv (D_k, 0)$. We assume that $F(D_1) > F(D_2) > \ldots > F(D_n)$. Then,

<u>Theorem 1</u>: a) There exists an infinite number of waves connecting $D_1 \to A$.
b) There exists an infinite number of waves connecting $D_2 \to A$.

It is natural to ask how many connections there must be from $D_k \to A$ for $k > 3$. We conjecture there may exist zero, a finite number, or an infinite number of such connections. While we do not know of a rigorous proof of this result, we shall indicate why we believe it is true after we outline the proof of Theorem 1.

Theorem 1a is proved using a shooting argument. The basic idea of the shooting argument was suggested to the author by Professor John Mallet-Paret. To set up the shooting argument we must first introduce some notation. Let P be the critical point of F immediatly to the right of A. Note that the F assumes a local minimum at $U = P$. Let ℓ be the ray, in the phase plane,

$U = P$, $V < 0$. For a given value of θ, let $A_{SE}^{\theta}(z)$ be the trajectory which satisfies:

$$\text{(a)} \quad \lim_{z\to+\infty} A_{SE}^{\theta}(z) = \bar{A},$$

(b) $A_{SE}^{\theta}(z)$ 'approaches' $\bar{A}$ from the quadrant $U > A$, $V < 0$.

Let $D_{SW}^{\theta}(z)$ be the trajectory that satisfies

$$\text{(a)} \quad \lim_{z\to-\infty} D_{SW}^{\theta}(z) = \bar{D}_1$$

(b) $D_{SW}^{\theta}(z)$ 'leaves' $\bar{D}_1$ into the quadrant $U < D_1$, $V < 0$.

Note that $A_{SE}^{0}(z)$ must intersect ℓ at some point. We denote the V coordinate of this point by γ_0. Hence, for θ sufficiently small, $A_{SE}^{\theta}(z)$ intersects ℓ at least once. Let $\gamma_A(\theta)$ be the V coordinate of the first place where $A_{SE}^{\theta}(z)$ intersects ℓ as $A_{SE}^{\theta}(z)$ is followed backwards starting at $\bar{A}$. Clearly, $\lim_{\theta\downarrow 0} \gamma_A(\theta) = \gamma_o$.

Let N be a fixed positive integer. If θ is sufficiently small then $D_{SW}^{\theta}(z)$ must intersect ℓ at least N times. Let $\gamma_D^N(\theta)$ be the V coordinate of the point where $D_{SW}^{\theta}(z)$ intersects ℓ for the N^{th} time. In [4] is proved that there exists positive constants θ_1 and θ_2 such that $\theta_1 < \theta_2$ and:

(a) $\gamma_A(\theta)$ and $\gamma_D^N(\theta)$ are continuous functions of θ for $\theta < \theta_2$,

(3.1) (b) $\gamma_A(\theta_1) < \gamma_D^N(\theta_1)$,

(c) $\gamma_A(\theta_2) > \gamma_D^N(\theta_2)$.

The reason that (3.1b) is true is because $F(A) > F(D_1)$. To prove (3.1c) one uses the fact that $\lim_{\theta \to +\infty} \gamma_A(\theta) = 0$.

We conclude from (3.1) that there exists some speed, say $\theta = \theta_o$, for which $\gamma_A(\theta_0) = \gamma_D^N(\theta_0)$. This corresponds to a connection, $D_1 \to A$, which winds around the phase plane N times. Since N is arbitrary there must exist an infinite number of $D_1 \to A$ connections.

We indicate why Theorem 1b is true with a specific example. Suppose that F is in the equivalence class of functions shown in Figure 1. Theorem 1a implies that there exists an infinite number of waves which connect $2 \to 3$. A simple shooting argument shows that there must exist a $3 \to 4$ connection. Using these facts we show that there exists an infinite number of $1 \to 3$ connections.

Consider the graph associated with this equivalence class of functions. It begins when θ is very large with the array $\frac{3124}{3124}$. Using Proposition 1 and the fact that there exists a $3 \to 4$ connection, we conclude that eventually a '3' must be next to the '4' on the top side of the array. For example, we may have the sequence of connections:

$$(3.2) \qquad \frac{3124}{3124} \xrightarrow{2\to 4} \frac{314}{31224} \xrightarrow{1\to 4} \frac{34}{312214} \xrightarrow{3\to 4} \frac{4}{3122134}\,.$$

Of course, other sequences of connections are possible, but we do know that, since there exists a $3 \to 4$ connection, the array must eventually be of the form $\frac{4}{3....34}$. The four dots represent some permutation of the numbers 1, 1, 2, and 2.

We now wish to use the fact that there exists an infinite number of $2 \to 3$ connections. To illustrate what has to happen suppose we start where we left off in (3.2) with the array $\frac{4}{3122134}$. The only way a $2 \to 3$

connection can take place is if a two is eventually next to the left-handed three on bottom side of the array. This cannot happen immediately because of the one separating the two and the left handed three. Therefore, there must eventually be a $1 \rightarrow 3$ connection. For example, we may have the sequence of connections:

$$\frac{4}{3122134} \xrightarrow{1\rightarrow 3} \frac{4}{3221134} \xrightarrow{2\rightarrow 3} \frac{4}{3211234} \xrightarrow{2\rightarrow 3} \frac{4}{3112234} .$$

It is not hard to show that, because there exists an infinite number of $2 \rightarrow 3$ connections, the array must equal $\frac{4}{3112234}$ an infinite number of times. But we know that there always has to be another $2 \rightarrow 3$ connection. Hence, a two has to be next to the left-handed three again, and there must be another $1 \rightarrow 3$ connection. Repeating this argument we conclude that there must exist an infinite number of $1 \rightarrow 3$ connections.

Recall that after the statement of Theorem 1 we conjectured that there exist functions for which there do not exist any waves which connect $D_3 \rightarrow A$. To understand why we believe this to be true consider the equivalence class of functions illustrated in Figure 8.

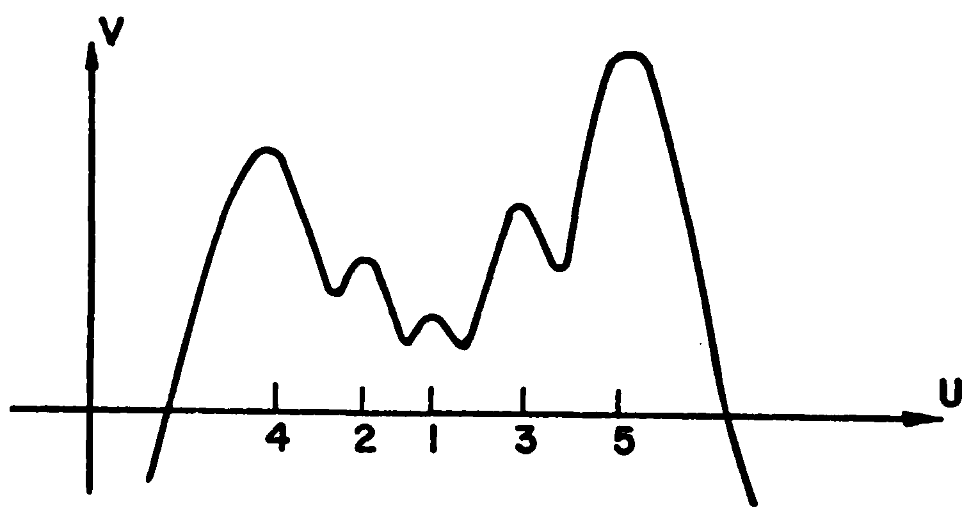

Figure 8

We claim that there does not have to exist any $1 \to 4$ connection. In (3.3) we show a path in the directed graph which does not contain any $1 \to 4$ connection. It does, however, contain an infinite number of $3 \to 4$, $2 \to 4$, $2 \to 3$, $1 \to 3$, and $1 \to 2$ connections which we know, from the previous results, must exist.

$$\frac{42135}{42135} \xrightarrow{1\to3} \frac{4235}{421135} \xrightarrow{\substack{3\to5\\2\to5\\4\to5}} \overset{*}{\frac{5}{421133245}} \xrightarrow{\substack{1\to2\\1\to2}} \frac{5}{423311245}$$

$$\xrightarrow{\substack{2\to4\\3\to4}} \frac{5}{431122345} \xrightarrow{\substack{1\to3\\1\to3}} \frac{5}{432211345} \xrightarrow{\substack{2\to3\\2\to3}} \frac{5}{431122345} \xrightarrow{\substack{1\to3\\1\to3}} \frac{5}{432211345} \tag{3.3}$$

$$\xrightarrow{3\to4} \frac{5}{422113345} \xrightarrow{2\to4} \overset{*}{\frac{5}{421133245}} \longrightarrow \cdots\cdot$$

Note that the array $\frac{5}{421133245}$ appears twice in (3.3). We have labelled these two arrays with the symbol '*'. Hence, we can just keep repeating the connections between the two *'s to obtain the desired path.

This, of course, is not a rigorous proof that a $1 \to 4$ connection may not exist, since there may not exist a specific function $F(U)$ which realizes the path shown in (3.3).

It is also possible (in fact, easier) to construct paths for which there exists an infinite number of $1 \to 4$ connections.

§4. Further Applications and the Tristable Equation in Detail

In this section we treat, in detail, the case when equation (1.1) has three stable rest states. We assume, throughout, that F assumes a local maximum precisely when U is equal to either A, B, or C, where $A < B < C$. In the previous section we showed that if $F(B) < F(A) < F(C)$, then there exists an infinite number of waves which connect $B \to A$. If $F(B) < F(C) < F(A)$ then, by symmetry, there exists an infinite number of waves which connect $B \to C$. There are essentually two other cases to consider. These are:

$$
(3.1)\quad
\begin{aligned}
&\text{(a)}\quad F(A) < F(C) < F(B)\\
&\text{(b)}\quad F(A) < F(B) < F(C)
\end{aligned}
$$

If (3.1a) is satisfied then there exists a unique wave connecting $A \to B$ and another connecting $C \to B$. There are no waves connecting $A \to C$ or $C \to A$. These facts are proved by considering the directed graph corresponding to the equivalence class of functions satisfying (3.1a). The directed graph is shown in Figure 9A. In Figure 9A we used the required ordering $A = 1$, $B = 3$, and $C = 2$.

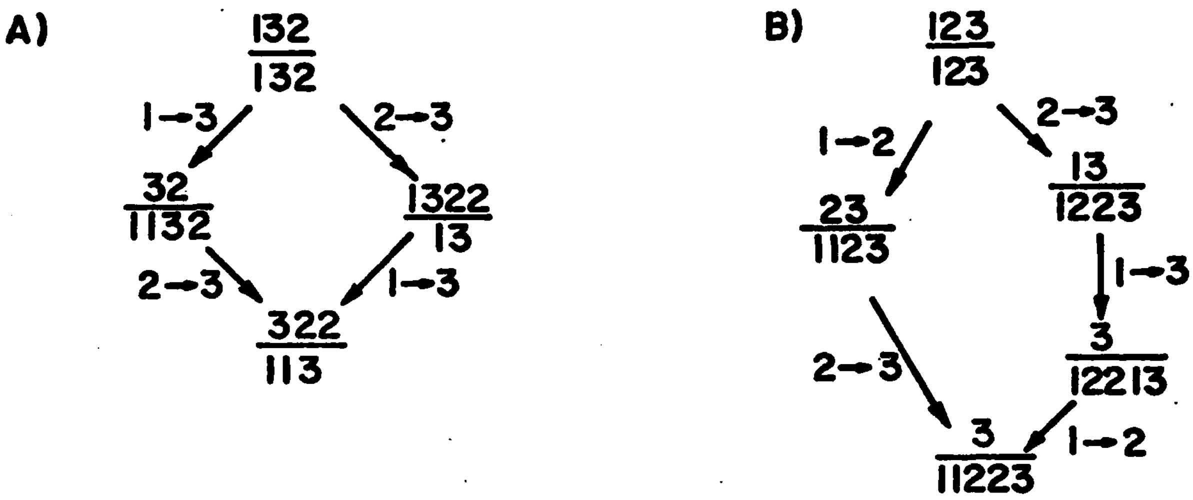

Figure 9. Directed graphs for functions with three local maxima. If these critical points are at U equal to A, B, and C, then in (A), F(A) < F(C) < F(B). In (B), F(A) < F(B) < F(C).

If (3.1b) is satisfied then there exists a unique wave connecting $A \to B$ and a unique wave connecting $B \to C$. There is a unique wave connecting $A \to C$ if and only if the wave connecting $A \to B$ is slower than the wave connecting $B \to C$. The directed graph for this equivalence class of functions is shown in Figure 9B. In Figure 9B we set $A = 1$, $B = 2$, and $C = 3$. This result is also proved in the paper of Fife and Mcleod [3].

Throughout the remainder of this section we assume that $F(B) < F(C)$. We wish to think of the number $F(A)$ as a bifurcation parameter. We have shown that if $F(A) > F(C)$ then there exists an infinite number of waves which connect $B \to C$, if $F(A) \in (F(B), F(C))$ then there exists an infinite number of waves which connect $B \to A$, and if $F(A) < F(B)$ then there exists only a finite number of waves. These waves are all illustrated in Figure 10. Figure 10 gives a qualitative description of which waves exist for a given value of the speed, θ, and another parameter, λ, which is related to the value of

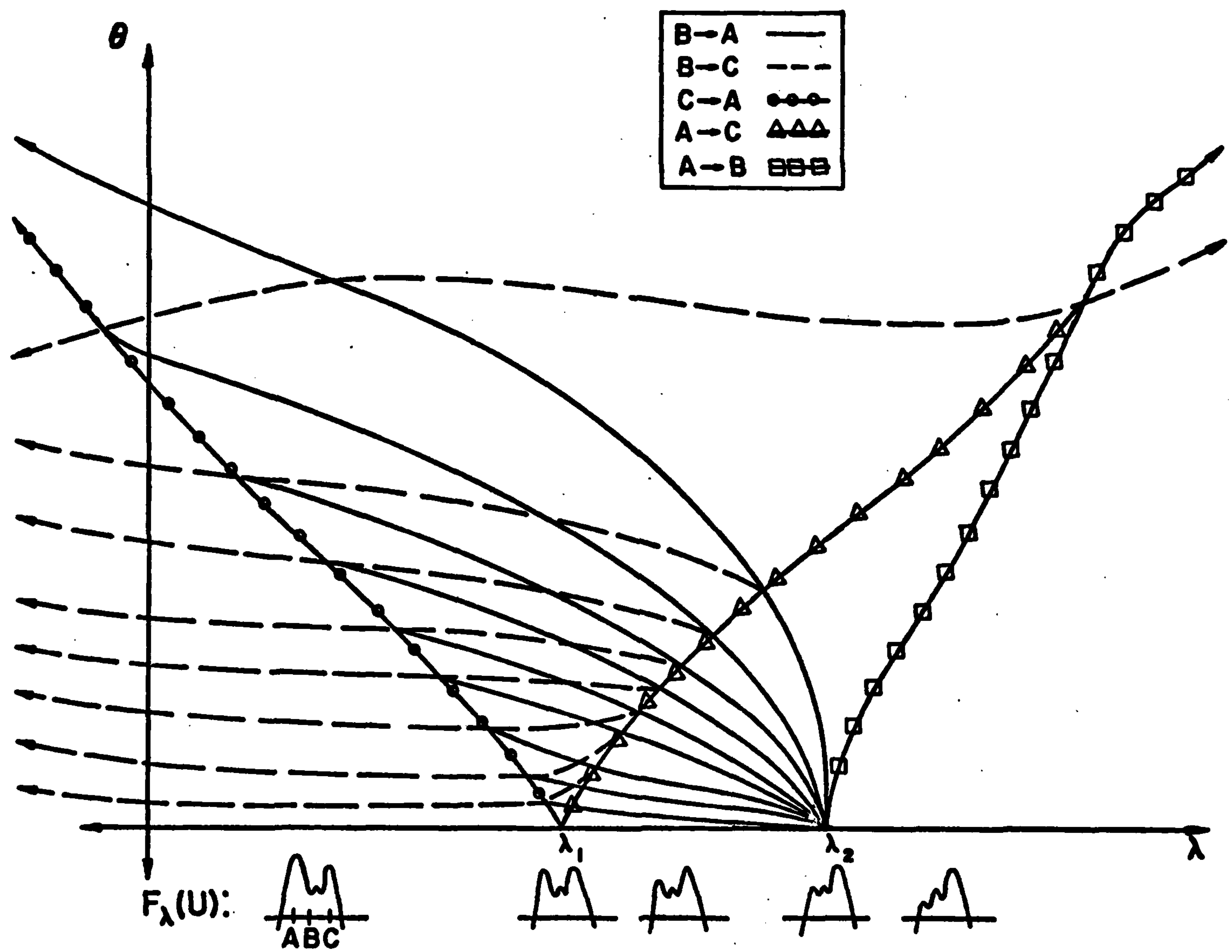

Figure 10. Global description of the tristable equation. It is assumed that for each value of λ, $F_\lambda(U)$ has a local maximum at $U = A$, B, and C. For $\lambda < \lambda_1$, $F(A) > F(C) > F(B)$. For $\lambda \in (\lambda_1, \lambda_2)$, $F(C) > F(A) > F(B)$, and for $\lambda > \lambda_2$, $F(C) > F(B) > F(A)$. There should be an infinite number of dashed curves ($B \to C$ connections) and solid curves ($B \to A$ connections). These curves converge to the axis $\theta = 0$.

$F(A)$. Let us denote the function $F(U)$ at a particular value of λ by $F_\lambda(U)$. As λ is increased, $F_\lambda(A)$ decreases. If $\lambda < \lambda_1$ then $F_\lambda(A) > F_\lambda(C)$, if $\lambda \in (\lambda_1, \lambda_2)$ then $F_\lambda(A) \in (F_\lambda(B), F_\lambda(C))$, and if $\lambda > \lambda_2$ then $F_\lambda(A) < F_\lambda(B)$. In Figure 10, the $B \to A$ connections are represented by solid curves, the $B \to C$ connections by dashed curves, the $C \to A$ connections by the solid curve with small circles, the $A \to C$ connections by the solid curve with small triangles, and the $A \to B$ connections by the solid curves with small squares. Note that there should be an infinite number of dashed curves, or $A \to B$ connections, and an infinite number of solid curves, or $B \to A$ connections. These two sets of curves are nested about the axis $\theta = 0$.

Figure 10 was drawn by considering how various phase planes change as the parameter $F(A)$ changes. Of course, the precise quantitative features of each curve in Figure 10 depends on exactly how the functions $F_\lambda(U)$ are chosen to vary with λ. Figure 10 does, however, illustrate the qualitative relationships described below between the various curves.

For each set of curves ($B \to A$ connections, $B \to C$ connections, etc...), the top, or fastest, curve corresponds to monotone traveling waves. These waves are asymptotically stable with respect to the partial differential equation (1.1). This was proved by Fife and McLeod [3]. The other curves in Figure 10 correspond to nonmonotone waves. For the dashed and solid curves ($B \to C$ and $B \to A$ connections) the n^{th} curve from the top, or the n^{th} fastest wave, corresponds to a connection which winds around $n-1$ times in the phase plane.

The most interesting points in Figure 10 are when (λ, θ) is equal to $(\lambda_1, 0)$ and $(\lambda_2, 0)$. When $\lambda = \lambda_1$ then $F(A) = F(C)$, and there exists waves with zero speed which connect $A \to C$ and $C \to A$. Note that when

$\lambda = \lambda_1$ there exists an infinite number of $B \to A$ and $B \to C$ connections. When $\lambda = \lambda_2$, $F(A) = F(B)$, and there exist waves with zero speed which connect $A \to B$ and $B \to A$. As λ is decreased from $\lambda = \lambda_2$, so that $F(B) < F(A)$, the infinite branches of $B \to A$ connections bifurcate from the point $(\lambda, \theta) = (\lambda_2, 0)$.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2417	2. GOVT ACCESSION NO. AD-A120 987	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) TRAVELING WAVE SOLUTIONS OF A MULTISTABLE REACTION-DIFFUSION EQUATION		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) David Terman		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 & MCS80-17158
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS * see below		12. REPORT DATE September 1982
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

* U.S. Army Research Office
P.O. Box 12211
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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Reaction-diffusion Equation, Traveling Wave Solution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new method for proving the existence of traveling wave solutions for equations of the form

$$u_t = u_{xx} + F'(u)$$

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20. ABSTRACT (cont.)

is presented. It is assumed that F is sufficiently smooth, $\lim_{|u|\to\infty} F(u) = -\infty$, F has only nondegenerate critical points, and if A and B are distinct critical points of F then $F(A) \neq F(B)$. The results describe when, for a given function F, there must exist zero, exactly one, a finite number, or an infinite number of waves which connect two fixed, stable rest points.